Anomalies

CFT's in even spacetime dimensions have "Weyl" anomalies

$\longrightarrow$ break conformal invariance in presence of background metric

$$\langle T^{\mu}{}_{\mu} \rangle = a \underbrace{E_d}_{\text{Euler density}} + \sum_i c_i \underbrace{I_i}_{\text{Weyl invariants}}$$

(recall $T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}$)

Unitary RG flows between CFTs in the UV and IR respect the following inequality

$$a_{UV} > a_{IR}$$

In 6d there are 3 different c_i with

$$c_{1,2,3} = c = 4 h^{\vee}_{\mathfrak{g}} d_{\mathfrak{g}} + r_{\mathfrak{g}}$$

As we will see $\Delta a = a_{\mathfrak{g}} - (a_{h} - 1) \sim b^2$

Anomaly polynomial:

When coupled to a background $SO(5)_R$ gauge field 1-form A, the anomaly polynomial is:

$$I_8(G) = r(G) I_8(1) + K(G) P_2(F)/24$$

where p_i are Pontryagin classes for the background $SO(5)_R$ field strength F :

$$p_1(F) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}\, F^2, \quad p_2(F) = \frac{1}{8}\left(\frac{i}{2\pi}\right)^4 \left((\mathrm{tr}\, F^2)\wedge(\mathrm{tr}\, F^2) - 2\,\mathrm{tr}\, F^4\right)$$

We have $p_1 = \lambda_1^2 + \lambda_2^2$, $p_2 = \lambda_1^2 \lambda_2^2$

where λ_1 and λ_2 are the Chern-roots of $F/2\pi$

I_8 is the anomaly polynomial 8-form, which is related to the anomalous gauge variation $I_6^{(1)}$ of the Lagrangian as follows:

$$I_8 = d I_7^{(0)}, \quad \underset{\substack{\uparrow \\ \text{gauge variation}}}{\delta} I_7^{(0)} = d I_6^{(1)}$$

$$I_8(1) = \Big(p_2(F) - p_2(\underset{\substack{\uparrow \\ \text{curvature} \\ \text{of background metric}}}{R}) + \frac{1}{4}\big(p_1(R) - p_1(F)\big)^2\Big)/48$$

$I_8(G)$ is the anomaly polynomial at the origin of tensor branch : $\Phi^I = 0 \quad \forall\, I$

It should be reproduced on the tensor branch as well !

Let us focus on branches in moduli space $\langle \Phi^I \rangle \neq 0$ describing a breaking pattern

$$G \to H \otimes U(1)$$
SCFT free tensor mult.

$U(1)$ theory needs a WZ-term to compensate for the difference in the R-symmetry anomaly

$$I_8(G) - I_8(H \otimes U(1)) = \frac{1}{24}\left(k(G) - k(H)\right) p_2(F)$$

for $\langle \Phi^I \rangle \neq 0$: $SO(5)_R \to SO(4)_R$

and moduli space of vacua

$$\mathcal{M}_c = SO(5)/SO(4) = S^4$$

with coordinates $\hat{\Phi}^I = \dfrac{\Phi^I}{r}$, $r = \left(\sum_{I=1}^{5} \Phi^I \Phi^I\right)^{\frac{1}{2}}$

This is compensated by the Wess Zumino term

$$S_{WZ} = \frac{1}{6}\left(c(G) - c(H)\right) \int_{\Sigma_7} \Omega_3(\hat{\Phi}, A) \wedge d\Omega_3(\hat{\Phi}, A) + \cdots$$

Σ_7 is a 7-dim space with boundary the 6d spacetime W_6 of the $\mathcal{N} = (2,0)$ theory:

$$\partial \Sigma_7 = W_6$$

$\Omega_3(\hat{\Phi}, A)$ is a 3-form defined as follows.

Consider the 4-form

$$\eta_4(\hat{\Phi}, A) \equiv \frac{1}{2} e_4^{\Sigma}$$

$$\equiv \frac{1}{64\pi^2} \varepsilon_{I_1 \cdots I_5} \Big[(D_{i_1}\hat{\Phi})^{I_1} (D_{i_2}\hat{\Phi})^{I_2} (D_{i_3}\hat{\Phi})^{I_3} (D_{i_4}\hat{\Phi})^{I_4}$$

$$- 2F_{i_1 i_2}^{I_1 I_2} (D_{i_3}\hat{\Phi})^{I_3} (D_{i_4}\hat{\Phi})^{I_4} + F_{i_1 i_2}^{I_1 I_2} F_{i_3 i_4}^{I_3 I_4} \Big] \hat{\Phi}^{I_5} dx^{i_1} \wedge \cdots \wedge dx^{i_4}$$

with $(D_i \Phi)^I = \partial_i \Phi^I - A_i^{IJ} \Phi^J$ with $I, J \in SO(5)_R$ and $A_i^{IJ} = -A_i^{JI}$ the background $SO(5)_R$ gauge field, and x^i coordinates on Σ_7.

$\eta_4(\hat{\Phi}, A=0) = \hat{\Phi}^*(\omega_4)$ is the pullback of the S^4 unit volume form, $\int_{S^4} \omega_4 = 1$.

Since $H^4(\Sigma_7) = 0$ and with $d\eta_4 = 0$ we get $\eta_4(\hat{\Phi}, A) = d\Omega_3(\hat{\Phi}, A)$

We have

$$d\Omega_3 \wedge d\Omega_3 = \frac{1}{4} e_4^{\Sigma} \wedge e_4^{\Sigma} = \frac{1}{4} p_2(F) + d\chi$$

where χ is invariant under $SO(5)_R$ gauge trf.

Using $d\Omega_3 \wedge d\Omega_3 = d(\Omega_3 \wedge d\Omega_3)$ and $p_2(F) = dp_2^{(0)}(A)$

we get for a $SO(5)_R$ gauge transformation:

$$\delta \int_{\Sigma_7} \Omega_3 \wedge d\Omega_3 = \frac{1}{4} \int_{\Sigma_7} \delta p_2^{(0)}(A) = \frac{1}{4} \int_{W_6} p_2^{(1)}(A)$$

where $p_2^{(1)}(A)$ is the anomaly 6-form by descent: $\delta p_2^{(0)} = dp_2^{(0)}$

Claim: $K_G = h_G^\vee d_G$

Example: $G = SU(N+1)$, $H = SU(N)$

$\rightarrow$ $K(G) = (N+1)^3 - (N+1)$, $K(H) = N^3 - N$

$\rightarrow$ WZ-term $= \frac{1}{2} N(N+1) \int_{\Sigma_7} \Omega_3(\hat{\Phi}, A) \wedge d\Omega_3(\hat{\Phi}, A)$

The 7-form $\Omega_3 \wedge d\Omega_3$ is not exact

$\rightarrow$ integral depends on choice of Σ_7:

$\Sigma_7 - \Sigma_7' \cong S^7$

$\rightarrow \frac{1}{6}\left(K(G) - K(H)\right) \int_{S^7} \Omega_3(\hat{\Phi}, A) \wedge d\Omega_3(\hat{\Phi}, A)$

for $A = 0$, the above integral is the Hopf number of the map $\hat{\Phi}^I : S^7 \rightarrow S^4$

$$\pi_7(S^4) = \mathbb{Z} + \mathbb{Z}_{12}$$

$\rightarrow \frac{1}{6}\left(\kappa(G) - \kappa(H)\right) \in \mathbb{Z}$

in order for $e^{2\pi i S}$ to be well-defined

Indeed $\kappa(G = SU(N)) = N^3 - N$ satisfies this!

<u>BPS strings:</u>

There are topologically stable, solitonic field configurations for $\langle \hat{\Phi}^I \rangle \neq 0$:

In d spacetime dimensions we have p-branes:

- field configurations $\hat{\Phi}^I(X_t)$ depending on transverse space X_t
- $\hat{\Phi}$ must approach a constant value when $X_t \rightarrow \infty$

$\rightarrow$ topologically classified by $\pi_{d-p-1}(\mathcal{M}_c)$

In the present case: $\mathcal{M}_c = S^4$, $d=6$

$\Rightarrow \pi_4(S^4) = \mathbb{Z}$

$\rightarrow$ non-trivial $p=1$ branes in $d=6$.

"solitonic strings" (M-strings)

topological charge: $N_s = \int_{X_t} \eta_4$

These strings act as flux sources for H_3 of the $U(1)$ $\mathcal{N}=(2,0)$ theory:

$$dH_3 = \alpha\, \eta_4$$

Action of strings: $S_{\text{string}} = \alpha \int_{W_6} B_2 \wedge \eta_4$

$\longrightarrow$ supersymmetry algebra
has central term: $Z = |Q\,\Phi|$
with $Q \sim N_s$ $\longrightarrow$ bound: $\underbrace{T}_{\text{tension}} \geq |Q\,\Phi|$

$\longrightarrow$ BPS field configurations
satisfying $T = |Q\,\Phi|$